

RATIONAL COMPLEXITY-ONE T -VARIETIES ARE WELL-POISED

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ABSTRACT. Given an affine rational complexity-one T -variety X , we construct an explicit embedding of X in affine space \mathbb{A}^n . We show that this embedding is well-poised, that is, every initial ideal of I_X is a prime ideal, and determine the tropicalization $\text{Trop}(X^\circ)$. We then study valuations of the coordinate ring R_X of X which respect the torus action, showing that for full rank valuations, the natural generators of R_X form a Khovanskii basis. This allows us to determine Newton-Okounkov bodies of rational projective complexity-one T -varieties, partially recovering (and generalizing) results of Petersen. We apply our results to describe all irreducible special fibers of $\mathbb{K}^* \times T$ -equivariant degenerations of rational projective complexity-one T -varieties, generalizing a results of Süß and the first author.

1. INTRODUCTION

Let $X \subset \mathbb{A}^n$ be an irreducible affine variety over a trivially valued, algebraically closed field \mathbb{K} . There is a corresponding presentation of the coordinate ring R_X of X by the polynomial algebra $\mathbb{K}[\mathbf{x}]$, where $\mathbf{x} = \{x_1, \dots, x_n\}$:

$$0 \longrightarrow I_X \longrightarrow \mathbb{K}[\mathbf{x}] \xrightarrow{\pi} R_X \longrightarrow 0.$$

Here I_X is the prime ideal of polynomials in $\mathbb{K}[\mathbf{x}]$ which vanish on X . We are concerned with the *tropical geometry* of the variety X . We consider the tropical variety $\text{Trop}(X^\circ) \subset \mathbb{R}^n$ associated to the very affine variety $X^\circ = X \cap (\mathbb{K}^*)^n$ ([MS15, Definition 3.2.1] and §3.1). We assume that $X^\circ \neq \emptyset$. Recall that to each point $w \in \text{Trop}(X^\circ)$ there is an associated initial ideal $\text{In}_w(I_X) \subset \mathbb{K}[\mathbf{x}]$. The purpose of this paper is in part to introduce the following property for affine embeddings:

Definition 1.1. The embedding of X into \mathbb{A}^n is said to be *well-poised* if $\text{In}_w(I_X)$ is a prime ideal for all $w \in \text{Trop}(X^\circ)$.

Well-poised embeddings can tell us much about the geometry of X . For example, when I_X is a homogeneous ideal, all of the Gröbner degenerations associated to the tropical points $w \in \text{Trop}(I_X)$ of a well-poised embedding are reduced, irreducible varieties. In particular, a generic tropical point defines a (possibly non-normal) toric degeneration of X .

Given the connection with toric geometry, it is not surprising that affine toric varieties themselves always have well-poised embeddings. It can be arranged that the coordinate ring of an affine toric variety X be presented by a prime binomial ideal J . With respect to this embedding, $\text{Trop}(X^\circ)$ is a rational vector space of dimension $\dim X$, and every initial ideal is J itself.

A more exotic example of a well-poised embedding is provided by the Plücker embedding of the Grassmannian variety of 2-planes:

$$\mathrm{Gr}(2, r) \subset \mathbb{P}^{\binom{r}{2}-1},$$

or rather, the embedding of the affine cone of this variety in $\mathbb{A}^{\binom{r}{2}}$. The tropical variety of this embedding is known as the tropical Grassmannian, and is described in the seminal paper [SS04] of Speyer and Sturmfels. The toric degenerations defined by the initial ideals of the tropical Grassmannian are also interesting objects. They have appeared in the study of integrable systems [NNU12] [HMM11], and invariant theory [HMSV09]. The variety $\mathrm{Gr}(2, 4)$ is an instructive special case. This variety is cut out by a single equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$ among the six Plücker variables p_{ij} , $1 \leq i < j \leq 4$. There are three ways to drop a monomial from this equation in order to make an irreducible binomial; these correspond to the three maximal cones of the tropical variety $\mathrm{Trop}(\mathrm{Gr}(2, 4)^\circ)$.

We can generalize the description of the case $\mathrm{Gr}(2, 4)$ to a special type of hypersurface. Consider a polynomial ring $\mathbb{K}[\mathbf{x}]$ and any partition $\{S_1, \dots, S_m\}$ of $[n]$. This defines an irreducible polynomial $f = \sum_{S_i} \prod_{j \in S_i} x_j$. Any binomial initial form extracted from this polynomial is likewise irreducible, so it follows that the embedded hypersurface cut out by f is well-poised. This last example is also interesting because when $m = 3$, the variety $V(f) \subset \mathbb{A}^n$ admits an effective action of an algebraic torus T of dimension one less than $\dim V(f)$.

In general, a complexity- k T -variety is a normal, irreducible variety X equipped with an action by an algebraic torus T such that $\dim T = \dim X - k$. By considering the action of a subtorus, any complexity- k T -variety can be viewed as a complexity- ℓ T -variety for $\ell > k$. In this sense, complexity-one T -varieties are a natural generalization of normal toric varieties, namely T -varieties of complexity-zero. Our first theorem generalizes the observation that affine toric varieties possess well-poised embeddings to a larger class of T -varieties:

Theorem 1.2 (Theorem 3.7). *Every affine rational complexity-one T -variety has an equivariant embedding $X \hookrightarrow \mathbb{A}^n$ which is well-poised.*

To prove this theorem, for any affine rational complexity-one T -variety X , we construct an explicit family of embeddings, which we call *semi-canonical embeddings*. Such an embedding is canonically determined by the geometry of X up to equivariant isomorphism and the action of $(\mathbb{K}^*)^n$.

The main idea behind the semi-canonical embedding is to embed X in such a fashion such that the geometry of X° becomes as simple as possible. Under this embedding $X \subset \mathbb{A}^n$, the intersection of X with $(\mathbb{K}^*)^n$ is simply the product of the torus T with $L^\circ = L \cap (\mathbb{K}^*)^m$, where L is a certain line in some projective space \mathbb{P}^m not contained in the boundary. Furthermore, the ideal I_X describing a semi-canonical embedding $X \subset \mathbb{A}^n$ can be determined from the combinatorial data describing the geometry of X and the ideal of L° . We describe this embedding in detail in §2.

Since the geometry of X° is especially simple, so is that of $\mathrm{Trop}(X^\circ)$: modulo lineality space, $\mathrm{Trop}(X^\circ)$ is just the tropical line $\mathrm{Trop}(L^\circ)$. In particular, we are able to describe a tropical basis for X° , see Corollary 3.5.

When an embedding of X is well-poised, work in [GRW16, Section 10] implies the existence of a section to the tropicalization map from the Berkovich skeleton

of X to its tropicalization. Our results here imply the existence of such a section over the tropicalization of any semi-canonical embedding. The presence of a large torus action gives this result a similar flavor to work of Draisma and Postinger in [DP16].

In the second half of the paper, we study the relationship between semi-canonical embeddings, higher rank valuations, and the theory of Newton-Okounkov bodies. For any full rank valuation \mathfrak{v} of the coordinate ring R_X , its image is a semigroup $S(R_X, \mathfrak{v})$. A set of generators of R_X is a *Khovanskii basis* if their images generate $S(R_X, \mathfrak{v})$, see §4.1. Our second main result shows the following:

Theorem 1.3 (Theorem 4.3). *Let X be an affine rational complexity-one T -variety, and \mathfrak{v} a full rank valuation which is homogeneous with respect to the grading induced by the T -action. Then X has a semi-canonical embedding for which the corresponding generators of R_X form a Khovanskii basis.*

This has a number of consequences. We are able to give explicit descriptions of the value semigroups $S(R_X, \mathfrak{v})$ for such valuations (Corollary 4.4). It turns out that all such valuations can be constructed as *weight valuations* using the machinery of Kaveh and the second author, see §4.3, Theorem 4.7, and [KM, §4].

Newton-Okounkov bodies of complexity 1 T -varieties have been studied by Petersen in [Pet], where he gives a construction of Newton-Okounkov polytopes for any projective, complexity-one T -variety Y with respect to valuations obtained from T -invariant flags. As a corollary of our results mentioned above, we recover Petersen's description in the case when Y is rational, and show that in fact, the Newton-Okounkov body with respect to *any* homogeneous valuation is of the form described by Petersen, see §4.5. Petersen also shows that the *global* Newton-Okounkov bodies for complexity-one T -varieties are polyhedral; we easily recover this result again in the case of rational T -varieties, see §4.6.

As a further application of our second main theorem, we determine all irreducible special fibers of $\mathbb{K}^* \times T$ -equivariant degenerations of rational projective complexity-one T -varieties, see Theorem 5.1. In [IS], Süß and the first author described all *normal* irreducible special fibers of such degenerations. Our theorem generalizes this result to allow for non-normal special fibers.

There is a large amount of literature devoted to studying the *algebra* of coordinate rings and ideals of toric varieties, for example, quadratic generation and Gröbner bases (e.g. [PRS98]), Koszulness (e.g. [BGT97]), and free resolutions (e.g. [Pee11]). We hope that the introduction of semi-canonical presentations for the coordinate rings of affine rational complexity-one T -varieties will lead to a similar study of the algebra of such T -varieties.

2. SEMI-CANONICAL EMBEDDINGS

2.1. T -Varieties. Let T be an algebraic torus, with character lattice M and co-character lattice N . Recall that a T -variety is a normal variety X equipped with an effective action $T \times X \rightarrow X$. The *complexity* of X is $\dim X - \dim T$. Such varieties may be described in terms of a 'quotient' variety Y of dimension equal to the complexity, equipped with some combinatorial data [AH06, AHS08, AIP⁺12].

We briefly survey this correspondence for affine T -varieties. Let Y be any normal variety, and fix a pointed, polyhedral cone σ in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. A *polyhedral divisor*

on Y with tailcone σ is a formal finite sum

$$\mathcal{D} = \sum_{i=0}^m \Delta_i \cdot P_i$$

where the P_i are distinct prime divisors on Y , and the coefficients Δ_i are polyhedra in $N_{\mathbb{Q}}$ with tailcone σ , unless $\Delta_i = \emptyset$. The polyhedral divisor \mathcal{D} induces a piecewise linear convex map

$$\begin{aligned} \mathcal{D} : \sigma^{\vee} &\rightarrow \operatorname{Div}_{\mathbb{Q} \cup \infty} Y \\ u &\mapsto \sum_i \Delta_i(u) \cdot P_i \end{aligned}$$

where

$$\Delta_i(u) := \min_{v \in \Delta_i} \langle v, u \rangle$$

and $\Delta_i(u) = \infty$ if $\Delta_i = \emptyset$. We use this to construct a T -scheme

$$X(\mathcal{D}) = \operatorname{Spec} \bigoplus_{u \in M \cap \sigma^{\vee}} H^0(Y, \mathcal{O}(\mathcal{D})).$$

Here an ∞ -coefficient in a divisor means that we allow poles with arbitrary order along the corresponding prime divisor. If one imposes certain positivity conditions on \mathcal{D} , $X(\mathcal{D})$ is actually a T -variety, and every affine T -variety can be constructed in this fashion [AH06, Theorems 3.1 and 3.4]. Note that in [AH06], these theorems are only stated when the characteristic of \mathbb{K} is zero. However, the proofs of these theorems should go through essentially verbatim in positive characteristic as well. We are only interested in the complexity-one case, which is dealt with explicitly in [Lan15].

In this paper, we are primarily concerned with rational, complexity-one, affine T -varieties. In this situation, we can take the quotient Y to be \mathbb{P}^1 , and the positivity condition on \mathcal{D} is exactly that $\sum \Delta_i \subsetneq \sigma$ [AH06, Example 2.12]. In the following, we will show how to embed such a variety X equivariantly in a particular toric variety. We will call this embedding a *semi-canonical embedding* of X . The ambient toric variety is uniquely determined by X up to equivariant isomorphism. The equations for X are not canonically determined, but their structure is.

2.2. Affine Embeddings. Suppose that we are given a rational, complexity-one affine T -variety X , which we have described as in §2.1 in terms of a polyhedral divisor

$$\mathcal{D} = \sum_{i=0}^m \Delta_i P_i$$

on \mathbb{P}^1 . As before, the Δ_i have common pointed tailcone σ (or equal \emptyset), and $\sum \Delta_i \subsetneq \sigma$. We will show how to embed $X = X(\mathcal{D})$ into an affine toric variety.

From the above data, we construct a pointed polyhedral cone in $N_{\mathbb{Q}} \times \mathbb{Q}^m$:

$$C = \mathbb{Q}_{\geq 0} \cdot (\{\Delta_i \times e_i\}_{i=0}^m \cup (\sigma \times 0))$$

where e_1, \dots, e_m is the standard basis of \mathbb{Q}^m , and $e_0 = -\sum_{i \neq 0} e_i$. Note that C is pointed, hence C^{\vee} is full-dimensional.

For any line $L \subset \mathbb{P}^m = \text{Proj } \mathbb{K}[y_0, \dots, y_m]$ intersecting the torus, let Q_i be the restriction to L of $V(y_i)$. We construct the following objects:

$$\begin{aligned} A(L) &= \bigoplus_{u \in \sigma^\vee \cap M} H^0 \left(L, \mathcal{O} \left(\sum_i [\Delta_i(u)] \cdot Q_i \right) \right) \cdot \chi^u \\ A(C) &= \mathbb{K}[C^\vee \cap (M \times \mathbb{Z}^m)] = \bigoplus_{(u,v) \in C^\vee \cap M \times \mathbb{Z}^m} \mathbb{K} \cdot \chi^{(u,v)} \\ X(L) &= \text{Spec } A(L) \quad X(C) = \text{Spec } A(C). \end{aligned}$$

Our complexity-one T -variety $X(\mathcal{D})$ is equivariantly isomorphic to $X(L)$ for an appropriate choice of L . Indeed, if we represent each point P_i by a choice of homogeneous coordinates $(p_{i0} : p_{i1})$, we can embed \mathbb{P}^1 into \mathbb{P}^m via

$$\begin{aligned} \phi : \mathbb{P}^1 &\rightarrow \mathbb{P}^m \\ (s_0 : s_1) &\mapsto (s_0 p_{00} - s_1 p_{01} : s_0 p_{10} - s_1 p_{11} : \dots : s_0 p_{m0} - s_1 p_{m1}). \end{aligned}$$

Taking $L_{\mathcal{D}}$ to be the image of ϕ in \mathbb{P}^m , the pullback to \mathbb{P}^1 of Q_i is exactly P_i . It follows by construction that $X(\mathcal{D})$ is equivariantly isomorphic to $X(L_{\mathcal{D}})$. Note that different choices $(p_{i0} : p_{i1})$ representing the points P_i correspond to acting on L by $(\mathbb{K}^*)^m$.

For any line $L \subset \mathbb{P}^m$, we will show that $X(L)$ comes with a natural closed embedding in the toric variety $X(C)$. Let $I(L)$ be the ideal of $L \cap (\mathbb{K}^*)^m$ in $(\mathbb{K}^*)^m$. We define $\tilde{I}(L)$ to be the ideal generated by

$$\{f \cdot \chi^u \mid f \in I(L) \text{ and } f \cdot \chi^u \in A(C)\}_{u \in M}.$$

Theorem 2.1. *There is a natural exact sequence*

$$0 \longrightarrow \tilde{I}(L)_u \longrightarrow A(C)_u \longrightarrow A(L)_u \longrightarrow 0$$

for every $u \in M \cap \sigma^\vee$, and hence an embedding of $X(L)$ in $X(C)$ with ideal $\tilde{I}(L)$.

Definition 2.2. Fix any rational affine complexity-one T -variety $X = X(\mathcal{D})$. For any choice of $L_{\mathcal{D}}$, a *semi-canonical embedding* of X is any affine embedding induced by the embedding $X(L_{\mathcal{D}}) \subset X(C)$, composed with an equivariant affine embedding of $X(C)$.

Proof of Theorem 2.1. It is straightforward to check that

$$A(C) = \bigoplus_{u \in \sigma^\vee \cap M} H^0 \left(\mathbb{P}^m, \mathcal{O} \left(\sum_i [\Delta_i(u)] \cdot Q_i \right) \right) \cdot \chi^u.$$

We thus let the map $A(C) \rightarrow A(L)$ be the map induced by restriction of sections. For degree $u \in M$, the degree u piece is just

$$H^0(\mathbb{P}^m, \mathcal{O}(D)) \longrightarrow H^0(L, \mathcal{O}_L(D))$$

where

$$D = \sum_i [\Delta_i(u)] \cdot V(y_i).$$

This is clearly surjective.

It remains to check that the kernel of this map is the degree u part of $\tilde{I}(L)$. Note that $f \cdot \chi^u \in A(C)$ if and only if

$$f \in H^0(\mathbb{P}^m, \mathcal{O}(D)).$$

Now, if we take such an f that also vanishes on $L \cap (\mathbb{K}^*)^m$, then clearly it restricts to zero on L , so $\tilde{I}(L)$ is contained in the kernel. On the other hand, sections

$$f \in H^0(\mathbb{P}^m, \mathcal{O}(D))$$

restricting to 0 on L are rational functions which vanish on L . Furthermore, since D is supported outside the torus, they are regular on $(\mathbb{K}^*)^m$. Hence, for such f , $f \cdot \chi^u \in \tilde{I}(L)$ as desired. \square

Example 2.3 (D_6 singularity). We consider X to be a normal surface singularity of type D_6 . This can be described by the polyhedral divisor

$$\mathcal{D} = \Delta_0 \cdot \{0\} + \Delta_1 \cdot \{1\} + \Delta_2 \cdot \{\infty\}$$

on \mathbb{P}^1 , where

$$\begin{aligned} \Delta_0 &= [3/2, \infty) \\ \Delta_1 &= [-1/2, \infty) \\ \Delta_2 &= [-1/2, \infty) \end{aligned}$$

The cone C is generated by the columns of

$$\begin{pmatrix} 3 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

and the dual cone is generated by the columns of

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

A Hilbert basis for $C^\vee \cap (M \times \mathbb{Z}^2)$ is given by the columns of

$$\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

This means that the toric variety $X(C)$ is cut out by a single binomial: $x_1^2 = x_2 x_3 x_4$. This corresponds to the linear relation between the above Hilbert basis elements.

Now, we can choose L so that the ideal I is generated by $f = \chi^{(1,0)} + \chi^{(0,1)} + 1$. To calculate $\tilde{I}(L)$, we check: for which (u, v_1, v_2) are the columns of

$$\begin{pmatrix} u & u & u \\ v_1 + 1 & v_1 & v_1 \\ v_2 & v_2 + 1 & v_2 \end{pmatrix}.$$

in C^\vee ? In this case, $\chi^{(2,2,1)} + \chi^{(2,1,2)} + \chi^{(2,1,1)}$ generates the ideal. Rewriting this in terms of variables corresponding to Hilbert basis elements, this becomes $x_2 + x_3 + x_4$. Hence, X is embedded in \mathbb{A}^4 with ideal $\langle x_1^2 - x_2 x_3 x_4, x_2 + x_3 + x_4 \rangle$.

Remark 2.4. If all points $Q_i \in L$ are distinct, the construction of $X(L)$ coincides with that of $X(\mathcal{D}')$, where \mathcal{D}' is the polyhedral divisor

$$\mathcal{D}' = \sum_i \Delta_i \cdot Q_i$$

on L . In particular, such $X(L)$ is normal.

On the other hand, if some points Q_i coincide, $X(L)$ may no longer be normal. We say that the collection of polyhedra $\{\Delta_i\}$ (with tailcone σ) is *admissible* for L if for every $Q \in L$ either

- (1) $\Delta_i = \emptyset$ for some $Q_i = Q$; or
- (2) for every $u \in \sigma^\vee$, $\Delta_i(u)$ is non-integral for at most one i with $Q_i = Q$.

Then just as in [IS, Proposition 4.2(iii)], $X(L)$ is normal if and only if $\{\Delta_i\}$ is admissible for L . See also [IV12, Corollary 2.6].

Remark 2.5. If L intersects the boundary in exactly two points, then $X(L)$ is a (potentially non-normal) toric variety. Indeed, after reordering suppose that Q_0, \dots, Q_k coincide, as do Q_{k+1}, \dots, Q_m . After indentifying these points respectively with $0, \infty \in \mathbb{P}^1$, it follows that $A(L)$ is the semigroup algebra for the semigroup

$$\left\{ (u, v) \in (\sigma^\vee \cap M) \times \mathbb{Z} \mid -\sum_{i \leq k} [\Delta_i(u)] \leq v \leq \sum_{i \geq k+1} [\Delta_i(u)] \right\}.$$

It is *a priori* not obvious that this is a finitely generated semigroup. However, by our embedding $X(L)$ in $X(C)$, we see not only that the semigroup is finitely generated, but are even given generators for it, namely, the restriction of generators for the semigroup of C^\vee .

2.3. The Ideal of $X(L)$. We may describe the ideal $\tilde{I}(L)$ quite explicitly. First of all, we have the following:

Lemma 2.6. *Let $\pi : \text{Spec } \mathbb{K}[M] \times (\mathbb{K}^*)^m \rightarrow (\mathbb{K}^*)^m$ be the projection. Then $\tilde{I}(L)$ is the ideal of the closure of $\pi^{-1}(L \cap (\mathbb{K}^*)^m)$ in $X(C)$.*

Proof. The ideal J of $\pi^{-1}(L \cap (\mathbb{K}^*)^m)$ in $\text{Spec } \mathbb{K}[M] \times (\mathbb{K}^*)^m$ is generated by the image of $I(L)$ in $\mathbb{K}[M \times \mathbb{Z}^m]$. The ideal J' of the closure in $X(C)$ consists of those $g \in J$ such that g is regular on $X(C)$, that is, $g \in A(C)$. Clearly, $\tilde{I}(L)$ is contained in J' .

On the other hand, $g \in J$ is in $A(C)$ if and only if each M -grade piece g_u is. But g_u is itself of the form $f \cdot \chi^u$ for some $f \in I(L)$, hence J' is contained in $\tilde{I}(L)$. \square

The above lemma means that the intersection $X^\circ = X \cap (\mathbb{K}^*)^n$ is just the product of $L^\circ = L \cap (\mathbb{K}^*)^m$ with the torus T .

We are now interested in finding generators for $\tilde{I}(L)$. We set $z_i = y_i/y_0$ for $i = 0, \dots, m$. Let $\mathcal{G}(L)$ be any set of generators for $I(L)$ which are linear in the z_i . The other ingredient we need is the polytope $P \subset M_{\mathbb{Q}} \times \mathbb{Q}^m$ consisting of those

(u, v) defined by the inequalities

$$\begin{aligned} v_i &\geq -\Delta_i(u) & i > 0; \text{ and} \\ \sum_{i=1}^m v_i &\leq \Delta_0(u) - 1. \end{aligned}$$

A set of *lattice generators for P as a C^\vee -module* is any set $\mathcal{P} \subset P \cap (M \times \mathbb{Z}^m)$ such that every lattice point of P is a sum of an element of \mathcal{P} with a lattice point of C^\vee .

Remark 2.7. A finite set \mathcal{P} of lattice generators for P may be computed by considering a Hilbert basis \mathcal{H} for the cone

$$\overline{\mathbb{Q}_{\geq 0}(P \times \{1\})}.$$

Selecting all elements of \mathcal{H} with final coordinate equal to one, and projecting back to $M \times \mathbb{Z}^m$ leads to such a generating set \mathcal{P} .

Proposition 2.8. *The ideal $\tilde{I}(L)$ is generated by $g \cdot z^v \cdot \chi^u$ for $g \in \mathcal{G}(L)$ and $(u, v) \in \mathcal{P}$.*

Proof. Fix a multidegree $u \in M$. For any Laurent monomial $z^v \in \mathbb{K}[z_1^{\pm 1}, \dots, z_m^{\pm 1}]$, $z^v \cdot \chi^u$ is in $A(C)$ if and only if

$$\begin{aligned} v_i &\geq -\Delta_i(u) & i > 0; \text{ and} \\ \sum_{i=1}^m v_i &\leq \Delta_0(u). \end{aligned}$$

Thus, the degree u piece of $A(C)$ is

$$g_u \cdot \chi^u \cdot \mathbb{K}[z_1, \dots, z_m]_{\leq d_u}$$

where

$$\begin{aligned} (1) \quad g_u &= z_1^{-\lfloor \Delta_1(u) \rfloor} \dots z_m^{-\lfloor \Delta_m(u) \rfloor} \\ d_u &= \sum_i \lfloor \Delta_i(u) \rfloor. \end{aligned}$$

We now note that the definition of P is such that $(u, v) \in P$ if and only if

$$z^v \cdot \chi^u, z_1 \cdot z^v \cdot \chi^u, \dots, z_m \cdot z^v \cdot \chi^u$$

are all in $A(C)$. In particular, for any $g \in \mathcal{G}(L)$, $g \cdot z^v \cdot \chi^u \in \tilde{I}(L)$ for each $(u, v) \in \mathcal{P}$.

We now need to show that these elements generate all of $\tilde{I}(L)$. Consider any homogeneous element in $\tilde{I}(L) \subset A(C)$, which we may write as $f \cdot g_u \cdot \chi^u$, with $f \in \mathbb{K}[z_1, \dots, z_m]_{\leq d_u}$. The polynomial f is in $I(L) \cap \mathbb{K}[z_1, \dots, z_m]$. Using that the $g \in \mathcal{G}(L)$ are linear and generate $I(L)$, we may write

$$f = \sum_{g \in \mathcal{G}(L)} c_g \cdot g$$

with all $c_g \cdot g \in \mathbb{K}[z_1, \dots, z_m]_{\leq d_u}$.

Hence, we have reduced to showing that we can generate any element of the form $g \cdot z^v \cdot \chi^u$ such that $g \cdot z^v \cdot \chi^u \in A(C)$, that is, $(u, v) \in P$. Now, any $(u, v) \in P$ can

be written as $(u', v') + (u'', v'')$, where $(u', v') \in \mathcal{P}$ and $(u'', v'') \in C^\vee$. Thus, we can generate $g \cdot z^v \cdot \chi^u$ as

$$g \cdot z^v \cdot \chi^u = z^{v''} \cdot \chi^{u''} \cdot (g \cdot z^{v'} \cdot \chi^{u'}),$$

noting that $z^{v''} \cdot \chi^{u''}$ is a regular function. \square

2.4. Projective T -Varieties. We are also interested in studying embeddings of projective T -varieties. Let X be a projective T -variety, and \mathcal{L} any ample line bundle. Then the ring

$$R(\mathcal{L}) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes i})$$

is a finitely generated normal domain, and $\text{Proj } R(\mathcal{L}) = X$. Choosing homogeneous generators for $R(\mathcal{L})$ as a \mathbb{K} -algebra of degrees d_0, d_1, \dots, d_n leads to a presentation

$$0 \longrightarrow I \longrightarrow \mathbb{K}[x_0, \dots, x_n] \longrightarrow R(\mathcal{L}) \longrightarrow 0$$

and hence an embedding of X in the weighted projective space $\mathbb{P}(d_0, d_1, \dots, d_n)$.

On the other hand, $\text{Spec } R(\mathcal{L})$ is an affine $T \times \mathbb{K}^*$ -variety. If we assume that X was rational and of complexity one, then so is $\text{Spec } R(\mathcal{L})$, and we have a semi-canonical presentation of $R(\mathcal{L})$ via the semi-canonical embedding $\text{Spec } R(\mathcal{L}) \subset X(C)$ of §2.2. Thus, after choosing the line bundle \mathcal{L} , we have a semi-canonical embedding of the polarized pair (X, \mathcal{L}) in some weighted projective space.

Example 2.9 (The projectivized cotangent bundle on \mathbb{P}^2). We consider $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$, the projectivization of the cotangent bundle on \mathbb{P}^2 . This is a Fano threefold often called W , and equal to number 2.32 in the list of Mori and Mukai [MM86]. It is also isomorphic to the variety of complete flags in \mathbb{K}^3 . This variety is equipped with a natural two-torus action.

On X , the anticanonical class is divisible by two. We take \mathcal{L} to be half the anticanonical bundle and consider the polarised pair (X, \mathcal{L}) . Using the data found in [Süs14], one determines that $\text{Spec } R(\mathcal{L})$ is encoded by the polyhedral divisor

$$\mathcal{D} = \Delta_0 \cdot \{0\} + \Delta_1 \cdot \{1\} + \Delta_2 \cdot \{\infty\}$$

on \mathbb{P}^1 , where the vertices of Δ_0 are $(1, 0, 0)$, $(0, 0, 0)$, the vertices of Δ_1 are $(0, 1, 0)$, $(0, 0, 0)$, the vertices of Δ_2 are $(0, 0, 1)$, $(-1, -1, 1)$, and all coefficients have tailcone generated by the columns of

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The distinguished \mathbb{K}^* -action on $\text{Spec } R(\mathcal{L})$ corresponds to the co-character $(0, 0, 1)$.

The dual cone C^\vee is generated by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

These elements also form a Hilbert basis for the semigroup $C^\vee \cap (M \times \mathbb{Z}^m)$. Note that all elements have degree one with respect to our \mathbb{Z} -grading, so we obtain an embedding of the Fano threefold X in \mathbb{P}^8 .

Concretely, the corresponding toric ideal is generated by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_4 & x_7 \\ x_1 & x_3 & x_8 \\ x_2 & x_5 & x_6 \end{pmatrix}$$

and we recognize that $X(C)$ is simply the (cone over the) Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. The variety X is further cut out by the additional equation $x_0 + x_3 + x_6 = 0$.

Remark 2.10. In general, the above construction produces a polarized pair (X, \mathcal{L}) in a (potentially weighted) projective space. If we wish to embed X in a standard projective space, we may then pass to the i th Veronese subring for appropriate choice of i . It is straightforward to check that the construction of §2.2 commutes with passing to Veronese subrings.

This means that the construction of 2.2 applied to $\text{Spec } R(\mathcal{L}^{\otimes i})$ will yield a semi-canonical embedding of the polarized pair $(X, \mathcal{L}^{\otimes i})$ in projective space.

Remark 2.11. One might also be interested in the total coordinate or *Cox ring* $\text{Cox}(X)$ of a rational, complexity-one projective T -variety X . See [ADHL15] for details on Cox rings.

As long as the class group of X is torsion-free (which is satisfied if e.g. X is smooth), then $\text{Spec Cox}(X)$ is itself a rational complexity-one affine T -variety, see [AP12] or [IV13] for a description of the corresponding polyhedral divisor. In any case, we may apply §2.2 to produce a semi-canonical presentation for the ring $\text{Cox}(X)$.

On the other hand, Hausen and Süß describe a presentation for the Cox ring of a rational complexity-one projective T -variety in [HS10, Theorem 1.3]. We believe that our presentation here agrees with theirs.

3. TROPICALIZATION AND GRÖBNER THEORY

3.1. Basics. Here we review the elements of Gröbner theory and tropical geometry necessary for our results. We recommend the books [CLO15, Stu96, MS15] as references. We will be working in the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$ and the corresponding ring of Laurent polynomials $\mathbb{K}[\mathbf{x}^{\pm 1}] = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Let $f = \sum c_{\alpha} x^{\alpha} \in \mathbb{K}[\mathbf{x}^{\pm 1}]$ and $w \in \mathbb{R}^n$. The *initial form* $\text{In}_w(f)$ of f with respect to w is

$$\text{In}_w(f) = \sum_{\substack{\alpha \\ \alpha \cdot w \text{ minimal}}} c_{\alpha} x^{\alpha}.$$

For an ideal I in $\mathbb{K}[\mathbf{x}]$ or $\mathbb{K}[\mathbf{x}^{\pm 1}]$, the *initial ideal* $\text{In}_w(I)$ is the ideal generated by the set $\{\text{In}_w(f) \mid f \in I\}$.

Definition 3.1. Let $X \subset \mathbb{A}^n$ be an affine variety intersecting the torus with ideal I , and I° the ideal of $X^{\circ} = X \cap (\mathbb{K}^*)^n$. The *tropicalization* $\text{Trop}(X^{\circ})$ of X° is the set of those $w \in \mathbb{R}^n$ such that $\text{In}_w(I^{\circ})$ does not contain a monomial.

Remark 3.2. It is straightforward to check that $\text{In}_w(I^{\circ})$ does not contain a monomial if and only if the same holds true for $\text{In}_w(I)$.

The tropical variety $\text{Trop}(X^{\circ})$ is typically given the structure of a polyhedral fan with the following property: if two elements $w, w' \in \text{Trop}(X^{\circ})$ are in the interior of the same face, then $\text{In}_w(I^{\circ}) = \text{In}_{w'}(I^{\circ})$. The structure theorem for tropical varieties [MS15, Theorem 3.3.6] says that $\text{Trop}(X^{\circ})$ has dimension $d = \dim X$ and

is connected in codimension 1. This means that any two maximal dimensional faces $C, C' \subset \text{Trop}(X^\circ)$ can be connected by a path which is contained in the relative interiors of d and $d - 1$ -dimensional cones.

Since we are interested in degenerations not of X° but of X , we consider a polyhedral structure so that if $w, w' \in \text{Trop}(X^\circ) \subset \mathbb{R}^n$ belong to the interior of the same cone then $\text{In}_w(I) = \text{In}_{w'}(I)$. This can be taken as a refinement of the above polyhedral structure, since it is straightforward to check that $\text{In}_w(I) = \text{In}_{w'}(I)$ implies $\text{In}_w(I^\circ) = \text{In}_{w'}(I^\circ)$.

In order to effectively compute the tropicalization of a variety, one desires a (finite) tropical basis:

Definition 3.3. A *tropical basis* for X° is a subset $G \subset I^\circ$ such that

$$\text{Trop}(X^\circ) = \bigcap_{f \in G} \text{Trop}(V(f)).$$

Every $X^\circ \subset (\mathbb{K}^*)^n$ has a finite tropical basis, see [MS15, Theorem 2.6.5].

3.2. Tropicalization of Semi-Canonical Embeddings. We now assume that we are in the situation of §2: we have an affine rational complexity-one T -variety $X = X(L)$ embedded in the toric variety $X(C)$ with ideal $\tilde{I}(L)$.

Fixing a generating set H of $C^\vee \cap (M \times \mathbb{Z}^m)$ of size n leads to an embedding of $X(C)$ and thus $X(L)$ into \mathbb{A}^n . We consider the polynomial ring

$$S = \mathbb{K}[x_u \mid u \in H].$$

Let $J(L)$ be the ideal in S of $X(L)$ under this embedding. We call the images of the x_u for $u \in H$ the *semi-canonical generators* of $A(L)$.

We are interested in describing $\text{Trop}(X(L)^\circ)$, where $X(L)^\circ = X(L) \cap (\mathbb{K}^*)^n$.

Proposition 3.4. *The tropicalization $\text{Trop}(X(L)^\circ)$ is the image of $N_{\mathbb{R}} \times \text{Trop}(L^\circ)$ in $\mathbb{R}^{\#H}$ under the injective linear map*

$$\begin{aligned} \phi : (N \oplus \mathbb{Z}^m)_{\mathbb{R}} &\rightarrow \mathbb{R}^{\#H} \\ v &\mapsto (\dots, \langle v, u \rangle, \dots)_{u \in H}. \end{aligned}$$

In particular, $\text{Trop}(X(L)^\circ)$ is the product of a tropical line with a plane of dimension $(\text{rank } N)$.

Proof. By construction, $X(L) \subset X(C)$, and $X(L)^\circ$ is the image in \mathbb{A}^n of $U := X(L) \cap \text{Spec } \mathbb{K}[M \times \mathbb{Z}^m]$. By Lemma 2.6, U is just $T \times L^\circ$, and the map $X(C) \rightarrow \mathbb{A}^n$ is given coordinate-wise by the functions χ^u for $u \in H$. Since the tropicalization of this monomial map is exactly ϕ , applying the tropicalization functor [MS15, Corollary 3.2.13] yields the claim. \square

Since the intersection points Q_i of L with the coordinate hyperplanes are all distinct, $\text{Trop}(L^\circ)$ is the one-dimensional fan in \mathbb{R}^m with rays $\rho_0, \rho_1, \dots, \rho_m$ generated by e_0, e_1, \dots, e_m . It follows that $\text{Trop}(X^\circ)$ has $m + 1$ maximal dimensional cones C_0, \dots, C_m , with $C_i = \phi(\rho_i \times N_{\mathbb{R}})$.

Corollary 3.5. *Let $\mathcal{G}(L)$ be a tropical basis for L° whose elements are dehomogenizations of linear forms. Then a tropical basis for $X(L)^\circ$ which generates the ideal $J(L) \subset S$ may be constructed by taking generators of the toric ideal of $X(C) \subset \mathbb{A}^n$, along with lifts to S of the generators of $\tilde{I}(L)$ produced by Proposition 2.8.*

Proof. After tropicalization, the generators of the toric ideal cut out the linear space given by the image of ϕ . It remains to cut out $\text{Trop}(X(L)^\circ)$ inside this linear subspace. The generators of $\tilde{I}(L)$ produced by Proposition 2.8 cut out $N_{\mathbb{R}} \times \text{Trop}(L^\circ)$, so their lifts to S will cut out $\text{Trop}(X(L)^\circ)$ after intersecting with $\text{Im } \phi$. \square

We also wish to describe $\text{In}_w(J(L))$ for $w \in \text{Trop}(X(L)^\circ)$. Since $\text{Trop}(X(L)^\circ)$ is contained in the image of the injective linear map ϕ above, weights $w \in \text{Trop}(X(L)^\circ)$ in this subspace can be interpreted as giving weights on monomials of $\mathbb{K}[M \times \mathbb{Z}^m]$. In particular, any $w \in \text{Trop}(X(L)^\circ)$ can be interpreted as a weight on $\mathbb{K}[\mathbb{Z}^m]$ or on the monomials of $A(C)$.

Definition 3.6. For any $w \in \text{Trop}(X(L)^\circ)$,

$$L_w := \overline{V(\text{In}_w I(L))} \subset \mathbb{P}^m.$$

The variety L_w is a line in \mathbb{P}^m . Furthermore, since $w \in \text{Trop}(X(L)^\circ)$, $\text{In}_w J(L)$ and thus $\text{In}_w I(L)$ does not contain a monomial. It follows that L_w is a line in \mathbb{P}^m intersecting the torus. We may thus apply the construction of §2 to obtain an irreducible variety $X(L_w)$ embedded in \mathbb{A}^n via an embedding in $X(C)$. Let $J(L_w)$ denote the ideal of this affine variety.

Theorem 3.7. For all $w \in \text{Trop}(X(L)^\circ)$,

$$\text{In}_w J(L) = J(L_w).$$

In particular, the embedding $X \hookrightarrow \mathbb{A}^n$ is well-poised.

Proof. Note that $J(L)$ is just the preimage of $\tilde{I}(L)$ in S . Likewise, $J(L_w)$ is the preimage of $\tilde{I}(L_w)$ in S . Hence, it will suffice to show that

$$\tilde{I}(L_w) = \text{In}_w(\tilde{I}(L))$$

with w interpreted as a weight in $A(C)$.

To that end, consider a homogeneous element $f \cdot \chi^u \in \tilde{I}(L)$, where $u \in M$ and $f \in I(L)$. Then

$$\text{In}_w(f \cdot \chi^u) = \text{In}_w(f) \cdot \chi^u.$$

Since $\text{In}_w(f) \in \text{In}_w(I(L))$, we obtain that $\text{In}_w(\tilde{I}(L)) \subset \tilde{I}(L_w)$.

The reverse inclusion follows from Proposition 2.8. Indeed, let $\mathcal{G}(L)$ be any set of generators of $I(L)$ linear in the z_i , whose initial terms with respect to w generate $\text{In}_w(I(L)) = I(L_w)$. By Proposition 2.8, $\text{In}_w(g) \cdot z^v \cdot \chi^u$ generate $\tilde{I}(L_w)$ for $g \in \mathcal{G}(L)$ and $(v, w) \in \mathcal{P}$. But by the same proposition, $g \cdot z^v \cdot \chi^u$ is then in $\tilde{I}(L)$. Hence, $\text{In}_w(\tilde{I}(L))$ generates $\tilde{I}(L_w)$. \square

Example 3.8. We continue Example 2.3. The tropicalization of X has three maximal cones. The corresponding initial ideals are (up to permutation of indices) all generated by $x_1^2 + x_2^2 x_3$. This corresponds to a degeneration to a non-normal toric variety.

Note that this embedding of $X(L)$ is not minimal: we can eliminate e.g. x_4 and arrive at the single equation $x_1^2 + x_2^2 x_3 + x_2 x_3^3 = 0$. This minimal embedding of $X(L)$ is not well-poised: one of the initial ideals is generated by $x_2^2 x_3 + x_2 x_3^3$, which is not prime.

Example 3.9. We continue Example 2.9 As in Example 3.8, the tropicalization has three maximal cones. The corresponding initial ideals are (up to permutation of indices) all generated by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_4 & x_7 \\ x_1 & -x_0 & x_8 \\ x_2 & x_5 & x_6 \end{pmatrix}$$

One may check directly that this initial ideal defines a normal toric variety. Alternatively, since the coefficients $\Delta_0, \Delta_1, \Delta_2$ are all lattice polytopes, Remarks 2.4 and 2.5 immediately imply that this is the case.

4. VALUATIONS AND NEWTON-OKOUNKOV BODIES

4.1. Basics. Let R be a \mathbb{K} -domain of finite type of Krull dimension d , and $(\Gamma, <)$ a totally ordered, finitely generated free abelian group. A \mathbb{K} -valuation of R with values in Γ is a map $\mathfrak{v} : R \setminus \{0\} \rightarrow \Gamma$ such that $\mathfrak{v}(\mathbb{K}^*) = 0$, for all $f, g \in R$, $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$, and $\mathfrak{v}(f+g) \geq \min\{\mathfrak{v}(f), \mathfrak{v}(g)\}$. For details on valuations, see [ZS75, Chapter VI]. The image of \mathfrak{v} is denoted $S(R, \mathfrak{v})$; it is an affine semigroup in Γ known as the *value semigroup*. We denote the closure of the positive hull of $S(R, \mathfrak{v})$ by $C(R, \mathfrak{v})$.

Any valuation \mathfrak{v} on R can be extended uniquely to a valuation on the field of fractions of R . Following standard terminology, the *rank* $r(\mathfrak{v})$ of \mathfrak{v} is the rank of the free Abelian group generated by $S(R, \mathfrak{v})$. This latter group is the same as the image of the extension of \mathfrak{v} to the field of fractions of R . The rank $r(\mathfrak{v})$ is always at most the Krull dimension of R , and we say that \mathfrak{v} is of *full rank* if equality holds. As \mathbb{K} is algebraically closed, if \mathfrak{v} has full rank then it follows that $\mathfrak{v}(f) = \mathfrak{v}(g)$ holds if and only if $\mathfrak{v}(f - cg) > \mathfrak{v}(f)$ for some $c \in \mathbb{K}^*$ for any $f, g \in R$ (see [KM, Theorem 2.4]). When this latter condition holds \mathfrak{v} is said to have *one dimensional-leaves*. For any valuation \mathfrak{v} with one-dimensional leaves, a *Khovanskii basis* for \mathfrak{v} is a subset of R whose image under \mathfrak{v} generates $S(R, \mathfrak{v})$.

4.2. Homogeneous Valuations on Complexity-One T -Varieties. Suppose that the domain R is graded by a lattice M . We say that a valuation \mathfrak{v} is *M -homogeneous* (or just homogeneous) if for all $f \in R$,

$$\mathfrak{v}(f) = \min_{u \in M} \{\mathfrak{v}(f_u)\}$$

where $f = \sum_{u \in M} f_u$ is the decomposition of f into graded pieces.

We will now study homogeneous valuations (and their value semigroups) for coordinate rings of affine rational complexity-one T -varieties. Indeed, let $X = X(L)$ be an affine rational complexity-one T -variety as in §2. The coordinate ring $R_X = A(L)$ is M -graded, and localizes to the ring $\mathbb{K}[L^\circ][M]$.

Fix a homomorphism $\psi : M \rightarrow \Gamma$, a point $Q \in L$, and an element $\gamma \in \Gamma$, $\gamma \geq 0$. The tuple (ψ, Q, γ) determines a homogeneous valuation $\mathfrak{v}_{(\psi, Q, \gamma)}$ of R_X by

$$f \cdot \chi^u \mapsto \psi(u) + \text{ord}_Q(f) \cdot \gamma$$

for $f \in \mathbb{K}(L)$, $u \in M$. Here, $\text{ord}_Q(f)$ is the order of vanishing of f at Q . It is straightforward to check that this indeed determines a valuation.

Proposition 4.1. *Every M -homogeneous valuation on R_X is of the form $\mathfrak{v}_{(\psi, Q, \gamma)}$.*

Proof. The data of a valuation on R_X is equivalent to the data of a valuation on $\mathbb{K}(L)[M]$. Homogeneous valuations on the latter are determined by their restrictions to $\mathbb{K}(L)$ and $\mathbb{K}[M]$. A homogeneous valuation on $\mathbb{K}[M]$ is simply a homomorphism $\psi : M \rightarrow \Gamma$. Finally, it is well-known that every valuation of $\mathbb{K}(L)$ is of the form $f \mapsto \text{ord}_Q(f) \cdot \gamma$ for some $Q \in L$ and some $\gamma \in \Gamma$, $\gamma \geq 0$. \square

Remark 4.2. The valuation $\mathbf{v}_{(\psi, Q, \gamma)}$ is of full rank if and only if ψ is of full rank and γ is linearly independent of the image of ψ .

Theorem 4.3. *Let $\mathbf{v} = \mathbf{v}_{(\psi, Q, \gamma)}$ for any $Q \in L \setminus L^\circ$. Then the valuations of the semi-canonical generators of $A(L)$ generate the value semi-group $S(R_X, \mathbf{v})$. In particular, they form a Khovanskii basis if \mathbf{v} has one-dimensional leaves.*

Proof. Consider any homogeneous element $f \cdot \chi^u \in A(L)$. The claims will follow if we can find a monomial in the semi-canonical generators whose valuation is the same as that of $f \cdot \chi^u$.

Without loss of generality, assume that $Q = Q_1$. Now, $f \cdot \chi^u \in A(L)$ implies that

$$v_i := \text{ord}_{Q_i} f \geq -\Delta_i(u)$$

and

$$\sum_{i=1}^m v_i \leq \Delta_0(u).$$

But the monomial $g = z^v \cdot \chi^u$ is then in $A(C)$, see the proof of Proposition 2.8. This monomial is the image in $A(C)$ of a monomial in the semi-canonical generators. Furthermore if \bar{g} is the image of g in $A(L)$, we have

$$\mathbf{v}(\bar{g}) = \mathbf{v}(\bar{z}^v) + \mathbf{v}(\chi^u) = \mathbf{v}(z_1^{v_1}) + \mathbf{v}(\chi^u) = \mathbf{v}(f) + \mathbf{v}(\chi^u) = \mathbf{v}(f \cdot \chi^u).$$

\square

Corollary 4.4. *With \mathbf{v} as above and $Q = Q_j$, the value semigroup $S(R_X, \mathbf{v})$ is the image of*

$$(2) \quad \left\{ (u, v) \in (\sigma^\vee \cap M) \times \mathbb{Z} \mid -\lfloor \Delta_j(u) \rfloor \leq v \leq \sum_{i \neq j} \lfloor \Delta_i(u) \rfloor \right\}$$

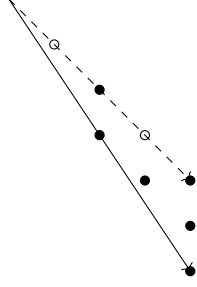
under the map $\rho : M \times \mathbb{Z} \rightarrow \Gamma$ given by $(u, \lambda) \mapsto \psi(u) + \lambda \cdot \gamma$. In particular, if \mathbf{v} has full rank, then $S(R, \mathbf{v})$ is isomorphic to the semigroup of (2).

Proof. It is straightforward to check that (2) is exactly the image of the monomials in the semi-canonical generators under the valuation $\mathbf{v}_{(\text{id}_M, Q_j, (0, 1))}$. The claim follows from the theorem, since

$$\mathbf{v} = \rho \circ \mathbf{v}_{(\text{id}_M, Q_j, (0, 1))}.$$

\square

Remark 4.5. The restriction in Theorem 4.3 and Corollary 4.4 that $Q \in L \setminus L^\circ$ is not a significant one: for any point $Q \in L$, one can re-embed L in a larger projective space so that the boundary points of this new line contain the original boundary of L together with Q . In particular, for any full rank homogeneous valuation on R_X , there is a semi-canonical embedding such that the semi-canonical generators are a Khovanskii basis.

FIGURE 1. Value semigroup for D_6 singularity

Example 4.6. We continue Examples 2.3 and 3.8 for X the D_6 singularity. Consider the valuation $\mathbf{v} = \mathbf{v}_{(\psi, Q_0, \gamma)}$ with $\psi : \mathbb{Z} \rightarrow \Gamma = \mathbb{Z}^2$ the inclusion of the first factor, and γ the second standard basis vector of \mathbb{Z}^2 . Then by Corollary 4.4, the value semigroup $S(R_X, \mathbf{v})$ equals

$$\left\{ (u, v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \mid \frac{-3u}{2} \leq v \leq 2 \left\lfloor \frac{-u}{2} \right\rfloor \right\}.$$

Generators are given by the valuations of the images of

$$\chi^3 y_1^2 y_2^2, \chi^2 y_1^1 y_2^2, \chi^2 y_1^2 y_2^1, \chi^2 y_1^1 y_2^1,$$

which are respectively $(3, -4)$, $(2, -3)$, $(2, -3)$, and $(2, -2)$, see Figure 1.

4.3. Connections to Tropical Geometry. Let $X \subset \mathbb{A}^n$ be an affine variety, and fix a presentation of its coordinate ring:

$$0 \longrightarrow I_X \longrightarrow \mathbb{K}[\mathbf{x}] \xrightarrow{\pi} R_X \longrightarrow 0.$$

In [KM, §3.4], Kaveh and the second author describe certain *weight valuations* $\mathbf{v}_W : R_X \setminus \{0\} \rightarrow \mathbb{Z}^r$ corresponding to a special choice of matrix W . The n columns of W are chosen in bijection with the variables \mathbf{x} from the group \mathbb{Z}^r taken with its standard lexicographic ordering. This choice defines a natural \mathbb{Z}^r -weighting of the monomials in $\mathbb{K}[\mathbf{x}]$, in particular \mathbf{x}^α is assigned $W\alpha \in \mathbb{Z}^r$. The value $\mathbf{v}_W(f)$ for $f \in R_X$ is then computed by the following formula:

$$\mathbf{v}_W(f) = \max\{\min\{W\alpha \mid p(\mathbf{x}) = \sum C_\alpha \mathbf{x}^\alpha, C_\alpha \neq 0\} \mid \pi(p(\mathbf{x})) = f\}.$$

One may also associate an initial ideal to the matrix W :

$$\text{In}_W(I_X) = \langle \text{In}_W(f) \mid f \in I \rangle$$

where for $f = \sum c_\alpha \mathbf{x}^\alpha$, $\text{In}_W(f)$ is the sum of monomial terms $c_\alpha \mathbf{x}^\alpha$ for which $W\alpha$ is minimal. This leads to the notion of *higher rank* tropical varieties:

$$\text{Trop}^r(I_X) = \{W \in \mathbb{Q}^{r \times n} \mid \text{In}_W(I_X) \text{ contains no monomials}\}.$$

Note that our definition of $\text{In}_W(I_X)$ extends naturally to matrices W with entries in \mathbb{Q} .

If R_1, \dots, R_r are the rows of W , [KM, Lemma 3.8 and Proposition 3.16] implies that:

$$(1) \text{In}_W(I_X) = \text{In}_{R_r}(\dots \text{In}_{R_1}(I_X) \dots),$$

(2) $W \in \text{Trop}^r(I_X)$ if and only if $R_i \in \text{Trop}(\text{In}_{R_{i-1}}(\dots \text{In}_{R_1}(I_X) \dots))$ for all i .

In general, the function \mathbf{v}_M is only a quasi-valuation (see [KM, §4]), however it defines a valuation of rank equal to the rank of W provided the associated initial ideal $\text{In}_W(I_X)$ is a prime ideal [KM, Lemma 4.4]. Under mild conditions, the columns of W are equal to the values of \mathbf{v}_W on the chosen generators of R_X ; this is in particular the case if W is a member of the higher rank tropical variety $\text{Trop}^r(I_X)$, see [KM, Proposition 4.6].

If $W \in \text{Trop}^r(I_X)$ and $\text{In}_W(I_X)$ is prime, then it follows from the above that the initial ideal $\text{In}_W(I_X)$ coincides with the initial ideal of a prime cone in $\text{Trop}(X^\circ)$. As the following theorem demonstrates, the situation is somewhat simplified for affine complexity-one T -varieties and their semi-canonical embeddings.

Theorem 4.7. *Let X be a rational affine complexity-one T -variety, and $\mathbf{v} : R_X \rightarrow \mathbb{Z}^r$ a full rank valuation. The following are equivalent:*

- (1) *The valuation \mathbf{v} is M -homogeneous;*
- (2) *A semi-canonical generating set is a Khovanskii basis for \mathbf{v} ;*
- (3) *The valuation \mathbf{v} equals \mathbf{v}_W for $W \in \text{Trop}^r(J(L))$ with $\text{rank}(W) = r$, where $J(L)$ is the ideal of forms vanishing on a semi-canonical generating set of $R_X = A(L)$.*

In the case that the above conditions hold, $\mathbf{v}|_{\mathbb{K}(L)} = \text{ord}_Q \cdot \gamma$ for some $\gamma \in \mathbb{Z}^r$ and some $Q = Q_i \in L \setminus L^\circ$. The initial ideal $\text{In}_W(J(L))$ coincides with the initial ideal of the facet $C_i \subset \text{Trop}(J(L))$. Furthermore, the value semigroup $S(R_X, \mathbf{v})$ is generated by the columns of W .

Proof. (1 \rightarrow 2) According to Proposition 4.1, $\mathbf{v}|_{\mathbb{K}(L)} = \text{ord}_Q \cdot \gamma$ for some $\gamma \in \Gamma$. We can re-embed L in a projective space so that the boundary points $L \setminus L^\circ$ contain Q . Theorem 4.3 completes the claim.

(2 \rightarrow 3) If a semi-canonical generating set $\mathcal{B} \subset A(L)$ is a Khovanskii basis, [KM, Proposition 2.18 and Lemma 4.10] implies that \mathbf{v} is a *subductive valuation*. This implies that $\mathbf{v} = \mathbf{v}_W$ for W a weight matrix of rank equal to the rank of \mathbf{v} .

(3 \rightarrow 1) Any element of a semi-canonical generating set is of the form $f\chi^u$ for $f \in \mathbb{K}(L)$ and $u \in M$. We may thus write $W = W_1 + W_2$, where the columns of W_1 are the values $\mathbf{v}_W(\chi^u)$ for $\chi^u \in \mathbb{K}[M]$ and the columns of W_2 are of the form $s \cdot \gamma$ for some fixed $\gamma \in \mathbb{Z}^r$ and $s = \text{ord}_Q(f)$ for $Q \in L \setminus L^\circ$. It follows that W_1 must have rank $r - 1$ and γ must be independent of the span of W_1 . Considering any $u \neq v \in M$, the above description of W implies that the values of two homogeneous elements $f\chi^u, g\chi^v$ are distinct; it follows that \mathbf{v}_W is M -homogeneous.

We now show the remaining claims. Note first that we must have $\text{In}_W(J(L)) = \text{In}_{R_r}(\dots \text{In}_{R_1}(J(L)) \dots)$. Let j be the index of the first non-zero entry of γ . Then the row R_j is of the form $\phi(w + s(\text{ord}_Q(y_1), \dots, \text{ord}_Q(y_m)))$ for some $s \neq 0$ and $w \in N$. It follows that $Q = Q_i$ for some i , otherwise $\text{ord}_Q(y_i) = 0$ for all i . Then $R_j \in C_i \subset \text{Trop}(J(L))$. Furthermore, each row $R_{j'}$ with $j' < j$ is in the lineality space of $J(L)$, and each row $R_{j'}$ with $j' > j$ is in the lineality space of $\text{In}_{R_j}(J(L))$; it follows that $\text{In}_W(J(L)) = \text{In}_{R_j}(J(L))$. \square

4.4. Graded Rings and Newton-Okounkov Bodies. Suppose that our finitely-generated \mathbb{K} -domain R is graded by $\mathbb{Z}_{\geq 0}$. Consider any homogeneous valuation $\mathbf{v} : R \setminus \{0\} \rightarrow \mathbb{Z} \times \Gamma$ such that for any homogeneous $f \in R$ of degree k , the first coordinate of $\mathbf{v}(f)$ is exactly $k \in \mathbb{Z}$. Then the *Newton-Okounkov Body* of R with

respect to \mathfrak{v} is

$$\Delta(R, \mathfrak{v}) = \pi_2(C(R, \mathfrak{v}) \cap \pi_1^{-1}(1))$$

where π_1, π_2 are the projections of $(\mathbb{Z} \times \Gamma)_{\mathbb{R}}$ to $\mathbb{Z}_{\mathbb{R}}$ and $\Gamma_{\mathbb{R}}$. This is a convex set, and under some mild hypotheses (including that $R_0 = \mathbb{K}$), it is compact, see e.g. [KK12].

Such a situation occurs when considering a divisor D on a projective variety X . Given any Γ -valued valuation \mathfrak{v} on $\mathbb{K}(X)$, one constructs a homogeneous valuation \mathfrak{v}_D on

$$R(\mathcal{O}(D)) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}(kD)).$$

as follows: for a section $s \in H^0(X, \mathcal{O}(kD))$ we have

$$\mathfrak{v}_D(s) = (k, \mathfrak{v}_{\mathcal{F}}(s)) \in \mathbb{Z} \times \Gamma.$$

The Newton-Okounkov body of D with respect to \mathfrak{v} is $\Delta(R(\mathcal{O}(D)), \mathfrak{v}_D)$.

For X a d -dimensional projective variety, a distinguished class of full rank valuations on $\mathbb{K}(X)$ comes from a choice of full flag $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_d = X$ of irreducible subvarieties of X , where the point \mathcal{F}_0 is a smooth point in each \mathcal{F}_i (see [LMta09, §1.1], [KK12, Example 2.13]). For $f \in \mathbb{K}(X)$ one computes the value $\mathfrak{v}_{\mathcal{F}}(f) = (a_1, \dots, a_d) \in \mathbb{Z}^d$ recursively, where \mathbb{Z}^d is endowed with the lexicographic ordering. The first component a_1 is taken to be the order of vanishing of f along the divisor \mathcal{F}_{d-1} . It follows that if s is a local equation for \mathcal{F}_{d-1} at p , $s^{-a_1}f$ can be regarded as a rational function on \mathcal{F}_{d-1} . This allows the procedure to be repeated with the divisor $\mathcal{F}_{d-2} \subset \mathcal{F}_{d-1}$ to produce a_2 , and so on until the process terminates with a_d , the order of vanishing at \mathcal{F}_0 . The flag \mathcal{F} is called a Parshin point of X , accordingly we refer to $\mathfrak{v}_{\mathcal{F}}$ as the Parshin valuation associated to \mathcal{F} .

In the case of such a Parshin valuation $\mathfrak{v}_{\mathcal{F}}$, it is customary to extend the valuation to a homogeneous valuation $\mathfrak{v}'_{\mathcal{F}, D}$ on $R(\mathcal{O}(D))$ as follows: if f_D is a local equation for D at the point \mathcal{F}_0 , for a section $s \in H^0(X, \mathcal{O}(kD))$ we have

$$\mathfrak{v}'_{\mathcal{F}, D}(s) = (k, \mathfrak{v}_{\mathcal{F}}(s) + k\mathfrak{v}_{\mathcal{F}}(f_D)) \in \mathbb{Z} \times \mathbb{Z}^d.$$

This is exactly the valuation considered in [LMta09]. It may be obtained from the valuation $(\mathfrak{v}_{\mathcal{F}})_D$ considered above by applying an invertible linear transformation. The advantage of this modified valuation $\mathfrak{v}'_{\mathcal{F}, D}$ is that The Newton-Okounkov body $\Delta(R(\mathcal{O}(D)), \mathfrak{v}'_{\mathcal{F}, D})$ depends only on the numerical equivalence class of D [LMta09, Proposition 4.1].

4.5. Newton-Okounkov Bodies for Rational Complexity-One T -Varieties.

Let X be a projective rational complexity-one T -variety and D a T -invariant divisor. Then $\text{Spec } R(\mathcal{O}(D))$ is an affine rational complexity-one $T \times \mathbb{K}^*$ -variety. The polyhedral divisor $\mathcal{D} = \sum_{i=0}^m \Delta_i P_i$ encoding it is described in [IS11, Example 2.5]. Note that in the rational complexity-one situation, the description there applies to arbitrary invariant divisors, not just very ample ones, after one applies the correction found in §6 of loc. cit. The polyhedral divisor \mathcal{D} (and its tailcone σ) live in $(\mathbb{Z} \times N)_D$; we set $\square_D = \{u \in M_{\mathbb{R}} \mid (1, u) \in \sigma^{\vee}\}$.

Now, if \mathfrak{v} is a valuation on $\mathbb{K}(X)$ which is homogeneous with respect to the grading of the character group M of T , then the valuation \mathfrak{v}_D on $R(\mathcal{O}(D))$ is $\mathbb{Z} \times M$ -homogeneous, and the results of §4.2 apply.

Theorem 4.8. *Let $\mathbf{v} = \mathbf{v}_{(\psi, Q, \gamma)}$ be an M -homogeneous valuation on $\mathbb{K}(X)$ with $Q = Q_j$. Then the Newton-Okounkov body of D with respect to \mathbf{v} is the image of*

$$(3) \quad \left\{ (u, v) \in \square_D \times \mathbb{R} \mid -\Delta_j(1, u) \leq v \leq \sum_{i \neq j} \Delta_i(1, u) \right\}$$

under the map $\rho : (M \times \mathbb{Z})_{\mathbb{R}} \rightarrow \Gamma_{\mathbb{R}}$ given by $(u, \lambda) \mapsto \psi(u) + \lambda \cdot \gamma$.

Proof. The valuation \mathbf{v}_D is the $\mathbb{Z} \times M$ -homogeneous valuation $\mathbf{v} = \mathbf{v}_{(\psi', Q, (0, \gamma))}$ with $\psi'(k, u) = (k, \psi(u))$ for $k, u \in \mathbb{Z} \times M$. The claim now follows from Corollary 4.4. \square

In [Pet], Petersen describes Newton-Okounkov bodies for invariant divisors on projective complexity-one T -varieties X with respect to Parshin valuations coming from T -invariant flags. Let $\pi : X \dashrightarrow \mathbb{P}^1$ be the rational quotient map, and let j be the largest index for which $\pi(\mathcal{F}_j)$ is a point $Q \in \mathbb{P}^1$. Then a local computation shows that $\mathbf{v}_{\mathcal{F}} = \mathbf{v}_{(\psi, Q, \gamma)}$ for some ψ, γ as in §4.2. Having thus determined Q , Theorem 4.8 allows us to describe $\Delta(D, \mathbf{v}_{\mathcal{F}})$, up to linear isomorphism. Comparing with [Pet, Theorem 3.9, Propositions 3.13 and 3.15], we see that we recover the same Newton-Okounkov bodies described by Petersen, up to linear isomorphism.

4.6. Global Newton-Okounkov Bodies. Let X be a smooth d -dimensional projective variety, and \mathcal{F} a Parshin point. Fix a basis D_1, \dots, D_r of divisors of $N^1(X)$, the group of divisors on X modulo numerical equivalence.

Definition 4.9. [LMta09, §4] The *global Newton-Okounkov body* $\Delta(X, \mathcal{F})$ of X is the closure in $\mathbb{R}^d \times \mathbb{R}^r$ of the positive hull of all points in $\mathbb{Z}^d \times \mathbb{Z}^r$ of the form

$$(\mathbf{v}'_{\mathcal{F}, D}(H^0(X, \mathcal{O}(D))), u),$$

where $u \in \mathbb{Z}^r$ and $D = \sum_i u_i D_i$.

The cone $\Delta(X, \mathcal{F})$ is a cone projecting onto the pseudo-effective cone of X . This cone $\Delta(X, \mathcal{F})$ is independent of the choice of basis of $N^1(X)$, and for any big class $\xi \in N^1(X)$, the Newton-Okounkov body of ξ (with respect to \mathcal{F}) is the fiber over ξ of this projection [LMta09, Theorem 4.5].

In [Pet, Theorem 5.1], Petersen shows that $\Delta(X, \mathcal{F})$ is rational polyhedral whenever X is a complexity-one T -variety. We readily recover that result here under the additional assumption that X is rational:

Theorem 4.10. *For any smooth projective rational complexity-one T -variety X and any Parshin point \mathcal{F} , the global Newton-Okounkov body $\Delta(X, \mathcal{F})$ is rational polyhedral.*

Proof. For a smooth projective rational complexity-one T -variety, $N^1(X) \cong \text{Pic}(X) \cong \text{Cl}(X)$, so we may take D_1, \dots, D_r as a \mathbb{Z} -basis of $\text{Cl}(X)$. The *Cox ring* of X is the graded ring

$$\text{Cox}(X) = \bigoplus_{u \in \mathbb{Z}^r} H^0 \left(X, \mathcal{O}(\sum_i u_i D_i) \right) \cdot \chi^u;$$

it is finitely generated, see e.g. [HS10]. In fact, $\text{Spec Cox}(X)$ is an affine, rational complexity-one T -variety, see Remark 2.11.

Then by construction, $\Delta(X, \mathcal{F})$ is just the cone $C(\text{Cox}(X), \mathbf{v})$ for the homogeneous valuation sending $s \cdot \chi^u$ to $(\mathbf{v}_{\mathcal{F}, D}(s), u)$, where $s \in H^0(X, \mathcal{O}(D))$ for $D = \sum u_i D_i$. But $C(\text{Cox}(X), \mathbf{v})$ is the closure of the positive hull of a finitely generated semigroup by Theorem 4.3. Hence, it is a polyhedral cone. \square

5. TEST CONFIGURATIONS AND DEGENERATIONS

Let X be a projective variety, and \mathcal{L} an ample line bundle on X . A *test configuration* for the pair (X, \mathcal{L}) is a \mathbb{K} -equivariant flat family \tilde{X} over \mathbb{A}^1 equipped with a relatively ample equivariant \mathbb{Q} -line bundle $\tilde{\mathcal{L}}$ such that

- (1) The \mathbb{K}^* -action on $(\tilde{X}, \tilde{\mathcal{L}})$ lifts the standard action on \mathbb{A}^1 ;
- (2) The general fiber is isomorphic to X , with $\tilde{\mathcal{L}}$ restricting to \mathcal{L} .

Such test configurations are used in the definition of K-stability and the study of the existence of Kähler-Einstein metrics on Fano manifolds, see e.g. [CDS14].

Suppose that the variety in question has an action by some algebraic group G . We say that a test configuration $(\tilde{X}, \tilde{\mathcal{L}})$ is G -equivariant if $(\tilde{X}, \tilde{\mathcal{L}})$ is equipped with a G -action extending the action on X , and commuting with the \mathbb{K}^* -action. In [IS, Proposition 4.2 and Theorem 4.3], Süß and the first author classified all normal irreducible special fibers appearing in test configurations for projective rational complexity-one T -varieties. This led to an effective criterion for determining the existence of a Kähler-Einstein metric on a Fano complexity-one T -variety [IS, Theorem 4.10].

The approach of [IS] to the classification of normal special fibers of test configurations utilized the fact that the total space of such a test configuration is itself a rational complexity-one T -variety. If one drops the normality assumption for the special fiber, then the total space must also no longer be normal and these methods do not apply. However, we can use our results on homogeneous valuations here to deal with non-normal special fibers:

Let (X, \mathcal{L}) be a polarized projective rational complexity-one T -variety, and let $L \subset \mathbb{P}^m$ and $\Delta_0, \dots, \Delta_m \subset (N \times \mathbb{Z})_{\mathbb{R}}$ be such that

$$X(L) \cong \text{Spec } R(\mathcal{L}).$$

Theorem 5.1. *For any T -equivariant test configuration \tilde{X} with irreducible special fiber X_0 , either $X_0 \cong X$ or X_0 is a toric variety of the form $\text{Proj } \mathbb{K}[S]$, where*

$$S = \left\{ (u, v) \in (\sigma^\vee \cap M \times \mathbb{Z}) \times \mathbb{Z} \mid -\lfloor \Delta_j(u) \rfloor \leq v \leq \sum_{i \neq j} \lfloor \Delta_i(u) \rfloor \right\}$$

for some $0 \leq j \leq r$, or

$$S = \left\{ (u, v) \in (\sigma^\vee \cap M \times \mathbb{Z}) \times \mathbb{Z} \mid 0 \leq v \leq \sum_i \lfloor \Delta_i(u) \rfloor \right\}.$$

Here, the \mathbb{Z} -grading for Proj is given by the first \mathbb{Z} -factor in $M \times \mathbb{Z} \times \mathbb{Z}$.

Proof. If X_0 is not isomorphic to X , then $T \times \mathbb{K}^*$ must act effectively on X_0 ; since X_0 is irreducible, it must be a toric variety.

By considering a sufficiently high multiple of $\tilde{\mathcal{L}}$, the test configuration \tilde{X} can be embedded equivariantly in some $\mathbb{P}^n \times \mathbb{A}^1$. If I is the ideal of X in \mathbb{P}^n , then the special fiber X_0 is given by the initial ideal $\text{In}_w(I)$ with regards to some weight w determined by the \mathbb{K}^* -action.

Since X_0 is reduced by assumption, $\text{In}_w(I)$ must be prime. In particular, it can only contain a monomial if it contains a variable, in which case I must have contained a linear form. In this case, X is contained in a smaller projective space, so after eventually passing to a linear subspace of \mathbb{P}^n , we can assume that $\text{In}_w(X)$

contains no monomials, that is, $w \in \text{Trop}(X^\circ)$. In fact, w is in a prime cone C of $\text{Trop}(X^\circ)$.

Let R be the homogeneous coordinate ring of X . Now, following [KM, Theorem 4], one can use the prime cone C to construct a homogeneous weight valuation \mathfrak{v} on R such that X_0 is Proj of the semigroup algebra for $S(R, \mathfrak{v})$. But since \mathfrak{v} is a homogeneous valuation, we may apply Theorem 4.3 for a concrete description of $S(R, \mathfrak{v})$. \square

Example 5.2. We continue Examples 2.9 and 3.9. Since all coefficients $\Delta_0, \Delta_1, \Delta_2$ are lattice polytopes, it follows from Theorem 5.1 that for any T -equivariant non-trivial test configuration of $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$ with irreducible special fiber, that special fiber is a *normal* toric variety. Up to isomorphism, only two possibilities occur. The first is the variety cut out by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_4 & x_7 \\ x_1 & -x_0 & x_8 \\ x_2 & x_5 & x_6 \end{pmatrix}$$

as described before. The second (corresponding to the case where \mathfrak{v} involves a point $Q \neq Q_i$) is cut out by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_4 & x_7 \\ x_1 & x_0 & x_8 \\ x_2 & x_5 & x_0 \end{pmatrix}.$$

We cannot see this second kind of degeneration as a Gröbner degeneration with regards to the embedding of Example 2.9, but must consider a semi-canonical embedding for X involving a fourth point on \mathbb{P}^1 apart from $0, 1, \infty$.

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REFERENCES

- [ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. *Cox rings*, volume 144 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.
- [AH06] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.
- [AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transform. Groups*, 13(2):215–242, 2008.
- [AIP⁺12] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The geometry of T -varieties. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 17–69. Eur. Math. Soc., Zürich, 2012.
- [AP12] Klaus Altmann and Lars Petersen. Cox rings of rational complexity-one T -varieties. *J. Pure Appl. Algebra*, 216(5):1146–1159, 2012.
- [BGT97] Winfried Bruns, Joseph Gubeladze, and Ngô Viêt Trung. Normal polytopes, triangulations, and Koszul algebras. *J. Reine Angew. Math.*, 485:123–160, 1997.
- [CDS14] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics and stability. *Int. Math. Res. Not. IMRN*, (8):2119–2125, 2014.
- [CLO15] David A. Cox, John Little, and Donal O’Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

- [DP16] Jan Draisma and Elisa Postinghel. Faithful tropicalisation and torus actions. *Manuscripta Math.*, 149(3-4):315–338, 2016.
- [GRW16] Walter Gubler, Joseph Rabinoff, and Annette Werner. Skeletons and tropicalizations. *Adv. Math.*, 294:150–215, 2016.
- [HMM11] Benjamin Howard, Christopher Manon, and John Millson. The toric geometry of triangulated polygons in Euclidean space. *Canad. J. Math.*, 63(4):878–937, 2011.
- [HMSV09] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil. The equations for the moduli space of n points on the line. *Duke Math. J.*, 146(2):175–226, 2009.
- [HS10] Jürgen Hausen and Hendrik Süß. The Cox ring of an algebraic variety with torus action. *Adv. Math.*, 225(2):977–1012, 2010.
- [IS] Nathan Ilten and Hendrik Süß. K-stability for Fano manifolds with torus action of complexity one. arXiv:1507.04442 [math.AG].
- [IS11] Nathan Owen Ilten and Hendrik Süß. Polarized complexity-1 T -varieties. *Michigan Math. J.*, 60(3):561–578, 2011.
- [IV12] Nathan Owen Ilten and Robert Vollmert. Deformations of rational T -varieties. *J. Algebraic Geom.*, 21(3):531–562, 2012.
- [IV13] Nathan Owen Ilten and Robert Vollmert. Upgrading and downgrading torus actions. *J. Pure Appl. Algebra*, 217(9):1583–1604, 2013.
- [KK12] Kiumars Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math. (2)*, 176(2):925–978, 2012.
- [KM] Kiumars Kaveh and Christopher Manon. Khovanskii bases, newton-okounkov polytopes and tropical geometry of projective varieties. arXiv:1610.00298 [math.AG].
- [Lan15] Kevin Langlois. Polyhedral divisors and torus actions of complexity one over arbitrary fields. *J. Pure Appl. Algebra*, 219(6):2015–2045, 2015.
- [LMta09] Robert Lazarsfeld and Mircea Mustață. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.
- [MM86] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 496–545. Kinokuniya, Tokyo, 1986.
- [MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [NNU12] Takeo Nishinou, Yuichi Nohara, and Kazushi Ueda. Potential functions via toric degenerations. *Proc. Japan Acad. Ser. A Math. Sci.*, 88(2):31–33, 2012.
- [Pee11] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.
- [Pet] Lars Petersen. Okounkov bodies of complexity-one T -varieties. arXiv:1108.0632 [math.AG].
- [PRS98] Irena Peeva, Victor Reiner, and Bernd Sturmfels. How to shell a monoid. *Math. Ann.*, 310(2):379–393, 1998.
- [SS04] David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.
- [Stu96] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
- [Süs14] Hendrik Süß. Fano threefolds with 2-torus action: a picture book. *Doc. Math.*, 19:905–940, 2014.
- [ZS75] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. Springer-Verlag, New York-Heidelberg, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.

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